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THE EXPLICIT INVERSE OF THE STIFFNESS MATRIX

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Abstract—In two recent papers the author has given the exact analytic expression of the internal forces in linear elastic structures composed of uniform prismatic elements. It was shown that the member forces are the ratios of two multilinear homogeneous polynomials in the unimodal stiffnesses of the elements of the structure. The order of the polynomials is equal to the number of nodal degrees of freedom of the structure. The number of terms of each polynomial is equal to the number of statically determinate stable substructures which can be derived from the original structure. The coefficients of the polynomials can be computed by employing the equilibrium equations and by enforcing global compatibility of deformations.

It was found empirically that the coefficients of the polynomial in the denominator were numerically equal to the square of the determinants of the statics matrices of the respective statically determinate substructures. As a consequence, the denominator became the sum of the stiffness matrices of the statically determinate substructures. This is in fact the Binet–Cauchy form of the determinant of the stiffness matrix of the structure. Bearing in mind that the inverse of the stiffness matrix can be expressed as the ratio of the adjoint matrix of the stiffness matrix divided by the determinant of the stiffness matrix it became clear that the expressions of the stress resultants stem from an explicit expression of the adjoint.

The explicit expression of the adjoint of the stiffness matrix lies at the heart of this paper. It is shown that the adjoint is a congruent transformation of the $(N-1)$ compound of the stiffness matrix, where N is the number of degrees of freedom of the structure. This cleared the way to use the Binet–Cauchy theorem on the product of compound matrices to obtain an explicit expression for the adjoint, and, *ipso facto*, for the inverse of the stiffness matrix. Having now the displacements of the structure, the expression of the stress resultants, which was obtained independently, emerges in a very elegant manner. The member forces in a structure can be expressed as the weighted sum of the member forces in all its determinate substructures, when subjected to the applied loads. The weighting factors are the ratios of the determinants of the stiffness matrices of the substructures, to the determinant of the stiffness matrix of the original structure.

Both the explicit inverse of the stiffness matrix and the expression of the internal forces in the structure are, at present, of a theoretical nature. The number of terms involved in the polynomials is simply excessive for common engineering structures. However, ongoing reseach may yield more applicable expressions to be used, for instance, in the field of automated design of structures.

The theory is illustrated with the explicit analysis of a stayed mast.

NOMENCLATURE

A_i	cross-sectional area of component i
B_k	coefficient of polynomial related to substructure k
C_N^M	number of combinations of N elements out of M
E_i	Young's modulus of component i
e	M-vector of component deformations
I_i	moment of inertia of component i
$\mathbf{\tilde{I}}$	signed inversion matrix (eqn 11)
$\mathbf{K}$	$N \times N$ stiffness matrix of structure
$\mathbf{K}_k$	$N \times N$ stiffness matrix of substructure k
$\mathbf{K}^{-1}$	inverse matrix of $\mathbf{K}$
$\lvert\mathbf{K}\rvert$	determinant of stiffness matrix
adj $\mathbf{K}$	adjoint matrix of $\mathbf{K}$
$\mathbf{K}^{(N-1)}$	$(N-1)$ compound matrix of $\mathbf{K}$
L_i	length of component i
M	number of (unimodal) components of structure
N	number of unconstrained nodal degrees of freedom
$\mathbf{p}$	N-vector of applied nodal loads
q_n	nth column of $\mathbf{Q}^{(N-1)}$
$\mathbf{Q}$	$N \times M$ statics matrix of structure
$\mathbf{Q}_k$	$N \times N$ statics matrix of substructure k
R	degree of statical redundancy of structure
$\mathbf{R}$	$M \times N$ kinematics matrix of structure
$\mathbf{R}_k$	$N \times N$ kinematics matrix of substructure k

s_i	unimodal stiffness of component i
$\mathbf{S}$	$M \times M$ diagonal matrix of stiffnesses of components of structure
$\mathbf{S}_k$	$N \times N$ diagonal matrix of stiffnesses of components of substructure k
t_j	internal force in component j of structure
t_{jk}	internal force in component j of substructure k
$\mathbf{t}$	M-vector of component forces in structure
$\mathbf{t}_k$	M-vector of component forces in substructure k
$\mathbf{u}$	N-vector of nodal displacements
π_k	product of stiffnesses of components of substructure k
π_n	entry (n, n) of $\mathbf{S}^{N-1}$

1. INTRODUCTION

Since the introduction of mathematical programming methods for the automated design of structures (Schmit, 1960), it has become clear that one needs explicit expressions for the displacements and internal forces in terms of the design variables. Indeed, mathematical programming techniques are basically sophisticated trial and error methods to optimize a constrained objective function. The algorithms usually navigate through a myriad of candidate design points until a satisfactory solution can be produced. Translated into structural terms, this calls for the analysis of all the candidate structures in order to verify whether they satisfy the constraints, a prospect unlikely to accommodate cost-effective minded engineers.

This state of affairs was the impetus for renewed interest in structural reanalyis and approximate analysis methods. A broad definition of structural reanalysis would encompass all the techniques which produce exact or approximate estimates of the structural response without performing a full analysis of the structure. Many ingenious methods have been developed over the years, several of which are quoted in the review papers of Arora (1976) and Abu Kassim and Topping (1985). The most prominent techniques nowadays are based on truncated, usually linear, Taylor series expansions of structural response quantities in terms of the cross-sectional design variables. With the exception of but a few, they are offsprings of the Reciprocal approximation. Initially developed for the nodal displacements of a truss in terms of the cross-sectional areas, Reciprocal type approximations proved valuable for more general structures, and they were also employed for the design of structures for optimal geometry.

Underlying all these efforts is the absence of the explicit inverse of the stiffness matrix. If we had the explicit expression of the inverse of the stiffness matrix, structural reanalysis, or simply structural analysis for that matter, would be confined to evaluating explicit expressions. The quest for the explicit inverse of the stiffness matrix in structural theory bears some resemblance to the quest for Eldorado (the golden one) by the Spanish conquistadors. The search has long since been called off. Structural engineers seem to have given up any hope of explicitly inverting the stiffness matrix. Having spent close to two decades of research in optimal structural design, this author, for one, has developed what may rightly be called an obsession with the inverse of the stiffness matrix. The effort has borne fruit, but unfortunately the explicit expressions in their present form are unusable. This is due to the immense number of terms involved in the equations. Nevertheless, the results are most interesting, and the way to reach the explicit inverse constitutes the subject matter of this paper.

The present theory is closely related to results published by Fuchs (1992a) which gave the analytic expression of the internal forces in a linear elastic redundant truss as a function of the axial stiffnesses of the elements. Assuming that the structure has M members and N nodal degrees of freedom ($M \geqslant N$), it was shown that the internal force in a bar j of a truss can be expressed analytically as the ratio of two multilinear polynomials of order N, in the axial stiffnesses of the structure

$$t_j = \frac{\sum_k B_k t_{jk} \pi_k}{\sum_k B_k \pi_k}, \quad j = 1, \ldots, M \tag{1}$$

where the summation index k is carried out over all the statically determinate stable substructures which can be derived from the redundant structure. The t_{jk}s are the internal forces in bar j if the external loads are applied to substructure k only, and the π_ks are the products of the stiffnesses of the N bars composing substructure k

$$\pi_k = s_i s_j \dots s_m \quad (N \text{ terms}) \tag{2}$$

where the typical axial stiffness $s_i = E_i A_i / L_i$, and E_i, A_i and L_i are respectively Young's modulus, the cross-sectional area and the length of element i. It was further shown that the coefficients B_k of the polynomials can be computed by enforcing overall compatibility of deformations.

As stated, the number of terms in the polynomials is equal to the number of statically determinate stable substructures which can be obtained from the original truss. The total number of combinations of N bars out of M is

$$C_N^M = \frac{M!}{N!R!} \tag{3}$$

where $R(=M-N)$ is the degree of static redundancy of the structure. It should be noted that this is an excessively high number for common engineering structures, even when we allow for the fact that many combinations result in unstable trusses.

In a subsequent paper (Fuchs, 1992b) the analytic expressions were extended to general structures composed of uniform prismatic elements, which can also carry bending and torsional moments. In this case, the products of stiffnesses in eqn (3) include axial stiffnesses, bending stiffnesses $E_i I_i / L_i$ and torsional stiffnesses $G_i J_i / L_i$ where I_i, G_i and J_i are respectively the moment of inertia (in both planes of symmetry), the shear modulus and the torsional rigidity of the cross-section of member i. In fact, these expressions can in principle be applied to all linear elastic finite element models.

Numerical experimentation with the analytic expressions has shown that the coefficients B_k in eqn (1) were consistently equal to the square of the determinants of the statics matrices, and therefore also equal to the products of the determinants of the statics and kinematics matrices of the corresponding substructures. Consequently every term in the denominator in eqn (1) is in fact the determinant of the stiffness matrix of the related statically determinate substructure. Based on the product of determinantal arrays which was found independently by Binet and Cauchy in 1812, it became manifest that the denominator in eqn (1) was nothing other than the determinant of the stiffness matrix of the structure.

Recalling that the inverse of a matrix can be expressed as the ratio of the adjoint of that matrix to its determinant, it was conjectured that the expression of the internal forces in the structure stem from a proto-expression of the inverse of the stiffness matrix. The missing link was an explicit formulation of the adjoint of the stiffness matrix. As will be shown in a subsequent section, the adjoint can be expressed explicitly by means of the theorem of product of compound matrices, also by Binet and Cauchy. It turns out that the adjoint of the stiffness matrix is the sum of generic matrices which are multiplied by products of combinations of $(N-1)$ unimodal stiffnesses of the type shown in eqn (2). Having found analytically the formulation of the inverse of the stiffness matrix and for that matter the nodal displacements, the expression of the internal forces in eqn (1) emerge naturally by premultiplying the nodal displacements with what is called in finite element nomenclature, the stress matrix.

As stated, the theory is hampered by an inordinate number of terms involved when applied to practical cases. It is therefore illustrated with a simple example: the explicit analysis of a stayed mast. The analytical expressions for this structure have a reasonable number of terms while allowing the reader to visualize the various aspects of the theory.

2. THE STRUCTURAL ANALYSIS EQUATIONS

This preamble is dedicated to writing the structural analysis equations in a form which will be useful for further developments. Consider a linear elastic truss of given geometry consisting of M members and N nodal degrees of freedom ($M \geqslant N$), which is subjected to an N-vector of static loads $\mathbf{p}$ applied at the nodes of the structure. The response of the structure is governed by the fundamental equations of structural theory:

$$\begin{array}{lll} (1) \text{ Statics:} & \mathbf{Qt} = \mathbf{p}; & \\ (2) \text{ Constitutive law:} & \mathbf{Se} = \mathbf{t}; & \\ (3) \text{ Kinematics:} & \mathbf{Ru} = \mathbf{e}; & (4) \end{array}$$

where $\mathbf{Q}$ is the $N \times M$ statics matrix, $\mathbf{t}$ is the M-vector of element axial forces, $\mathbf{S}$ is the $M \times M$ diagonal natural stiffness matrix ($S_{ij} = \delta_{ij} s_j$), δ_{ij} is the Kronecker delta, $\mathbf{e}$ is the M-vector of element total elongations, $\mathbf{R}(=\mathbf{Q}^T)$ is the $M \times N$ kinematics matrix and $\mathbf{u}$ is the N-vector of nodal displacements.

Similar equations can be written for structures composed of other types of elements such as beams, shear panels and plates. What sets the analysis equations of the truss apart from all other structures is the fact that the constitutive equations have a diagonal unassembled stiffness matrix S. When, for instance, bending elements are present in a structure, the stiffness matrix has 2×2 bending matrices along its diagonal, which relate the two end-moments of the elements to the two end-rotations with respect to the chord. Using the terminology of Fuchs (1991) we say that a truss element is unimodal and a beam element is bi-modal. The deformation of the latter is characterized by two quantities whereas the deformation of the truss elements is given by one quantity, its axial elongation. If we consider a 4-node plain strain element for example, its constitutive law has a 5×5 natural stiffness matrix. Its deformation pattern can be described by five quantities. For reasons which will become evident later on, it is useful to have a diagonal stiffness matrix in eqn (4).

As in Fuchs (1991) in the case of bending, the element stiffness matrix can be diagonalized by means of a modal analysis and by a congruent transformation based on a modal matrix. It was shown that the beam element is thus structurally equivalent to two unimodal elements mounted in parallel, a moment and a shear element. The moment element deforms symmetrically and carries the average bending moment in pure bending. The shear element deforms in an antisymmetric mode and carries the differential moments and related shear forces in "pure" shear. Similarly, the 4-node plane strain element is equivalent to five unimodal components mounted in parallel between the four nodes; two flexural, a shear, a stretching and a uniform extension component (Bathe and Wilson, 1976).

Consequently, in modal coordinates, every element can be represented by its unimodal components, which results in a diagonal natural stiffness matrix in eqns (4). Selection of statically determinate stable substructures from the original redundant one, can be achieved by choosing arbitrarily submatrices of rank N in the statics matrix $\mathbf{Q}$. The physical counterpart of this procedure is that substructures are obtained by jettisoning the redundant components, much in the same way as is done in the case of trusses. In conclusion, eqns (4) are valid for all linear elastic finite element models, including the property that the unassembled stiffness matrix S is strictly diagonal.

3. THE INVERSE OF THE STIFFNESS MATRIX

For a mathematical background on matrices and determinants the reader is referred to Aitkin's (1956) excellent book on the subject. This section and the following one draw liberally on that source.

The inverse of the $N \times N$ stiffness matrix $\mathbf{K}$ of a structure can be written as the adjoint matrix of $\mathbf{K}$ divided by the determinant of $\mathbf{K}$

$$\mathbf{K}^{-1} = \frac{\text{adj } \mathbf{K}}{|\mathbf{K}|}. \tag{5}$$

The adjoint matrix of $\mathbf{K}$ is an $N \times N$ matrix whose elements K_{ij} are the minors obtained by crossing out row i and column j of $\mathbf{K}$, multiplied by $(-1)^{i+j}$. Referring to the analysis equations (1), the stiffness matrix $\mathbf{K}$ can be expressed as the product of the statics matrix, the natural stiffness matrix and the kinematics matrix

$$\mathbf{K} = \mathbf{QSR}. \tag{6}$$

Note the statics and kinematics matrices are rectangular matrices of order $N \times M$ and $M \times N$ respectively. The stiffness matrix is a square matrix of order $N \times N$.

The expression for $|\mathbf{K}|$ is straightforward. Binet and Cauchy have shown in 1812 that the determinant of such a product of rectangular matrices can be obtained by selecting all the $N \times N$ matrices in $\mathbf{Q}$ and their corresponding matrices in $\mathbf{S}$ and $\mathbf{R}$ and summing up the product of their determinants. A typical submatrix $\mathbf{Q}_k$ is for instance composed of columns $ij\ldots m$ (N columns) of $\mathbf{Q}$. The corresponding matrix $\mathbf{S}_k$ will include rows $ij\ldots m$ and columns $ij\ldots m$ of $\mathbf{S}$. Similarly $\mathbf{R}_k$ is formed by rows $ij\ldots m$ of $\mathbf{R}$. The product of the determinants of these three matrices is a typical term in the expression of $|\mathbf{K}|$

$$|\mathbf{K}| = \sum_k |\mathbf{Q}_k||\mathbf{S}_k||\mathbf{R}_k|. \tag{7}$$

Noting that the determinant of a product of square matrices is equal to the product of the determinants of the matrices, and since $\mathbf{Q}_k\mathbf{S}_k\mathbf{R}_k$ is the stiffness matrix $\mathbf{K}_k$ of substructure k, eqn (7) can also be written as

$$|\mathbf{K}| = \sum_k |\mathbf{K}_k|. \tag{8}$$

Since unstable combinations of N structural elements yield zero $|\mathbf{Q}_k|$ determinants, we have the property that the determinant of the stiffness matrix of a structure is equal to the sum of the determinants of the stiffness matrices of all the stable substructures which can be derived from the structure.

Also, $\mathbf{S}_k$ being diagonal, its determinant is equal to the product of the diagonal entries of the matrix. This in conjunction with the property that the determinant of a matrix is equal to the determinant of the transpose of that matrix allows us to write eqn (7) in the alternative form

$$|\mathbf{K}| = \sum_k |\mathbf{Q}_k|^2 \pi_k. \tag{9}$$

The right-hand side of the above expression is reminiscent of the denominator of the analytic formulation of the internal loads of the structure [eqn (1)]. It suffices to set $B_k = |\mathbf{Q}_k|^2$ to get identical expressions. The remaining step in computing the inverse is to generate an expression for the adjoint of the stiffness matrix.

4. THE ADJOINT OF THE STIFFNESS MATRIX

Unlike the determinant of a product of rectangular matrices there is no straightforward expression for the adjoint of a product of rectangular matrices. There is however a Binet–Cauchy formula for the compound of a product of matrices, and as will be shown, there is a way to relate the $(N-1)$ compound of an $N \times N$ matrix to the adjoint of that matrix.

The compound of a matrix is a rather esoteric construct, and the safest way to describe it is probably to quote Aitkin's definition, verbatim (pp. 90–91): "Let a matrix be formed the elements of which are minors of $|\mathbf{A}|$ of order k; let all minors which come from the same group of k rows (or columns) of $\mathbf{A}$ be placed in the same row (or column) of this derived matrix; and let the priority of elements in rows or columns of this matrix be decided on the principle by which words are ordered in a dictionary. ... The matrix with elements minors of order k constructed in this way will be called the kth compound of $\mathbf{A}$ and will be denoted by $\mathbf{A}^{(k)}$. It will be defined in the same way even when $\mathbf{A}$ is rectangular of order $m \times n$. ... The order of $\mathbf{A}^{(k)}$ will then be $C_k^m \times C_k^n$."

We will start by expressing the adjoint of the stiffness matrix $\mathbf{K}$ in terms of the $(N-1)$ compound of $\mathbf{K}$. Both the adjoint and the $(N-1)$ compound are matrices of order $N \times N$ and are composed of elements which are minors of order $(N-1)$ of $\mathbf{K}$. They differ in two respects. In the first instance, the rows and columns of both matrices are in inverse order; that is, rows (columns) $12 \ldots N$ of the adjoint are the rows (columns) $N \ldots 21$ of the compound matrix. Consider for instance element $(1, 1)$ of the adjoint. It is the determinant obtained from $\mathbf{K}$ by suppressing the first row and the first column (the sign will be discussed shortly). However, according to the definition of the compound matrix, this is element (N, N) of the $(N-1)$ compound of $\mathbf{K}$.

The two matrices also differ by the sign of their elements. The adjoint is composed of signed minors, or cofactors, whereas the compound is populated by (unsigned) minors. The sign of element (i, j) of the adjoint is $(-1)^{i+j}$. Bearing in mind these two discrepancies it is easy to verify that the adjoint can be obtained by pre- and post-multiplication of the $(N-1)$ compound with the signed inversion matrix $\tilde{\mathbf{I}}$

$$\operatorname{adj} \mathbf{K} = \tilde{\mathbf{I}}\mathbf{K}^{(N-1)}\tilde{\mathbf{I}}^{\mathrm{T}} \tag{10}$$

where the inversion matrix $\tilde{\mathbf{I}}$ has, alternately, the values 1 and -1 on its secondary diagonal

$$\tilde{\mathbf{I}} = \begin{bmatrix} & & & & & 1 \\ & & & & -1 & \\ & & & 1 & & \\ & & -1 & & & \\ & \cdot & & & & \\ \cdot & & & & & \end{bmatrix}. \tag{11}$$

We now have to find an expression for the $(N-1)$ compound of the stiffness matrix. To do so we will use the product form of $\mathbf{K}$ (eqn 6) and rely again on Binet and Cauchy by using the beautiful theorem on the product of compound matrices: the k-compound of a product of matrices is equal to the product of the k-compounds of these matrices, in the same order. This theorem also holds for rectangular matrices. For the problem at hand the theorem yields

$$\mathbf{K}^{(N-1)} = \mathbf{Q}^{(N-1)}\mathbf{S}^{(N-1)}\mathbf{R}^{(N-1)} \tag{12}$$

which in conjunction with eqns (10) and (6) yields the Binet–Cauchy form of the adjoint of the stiffness matrix

$$\operatorname{adj} \mathbf{K} = \tilde{\mathbf{I}}\mathbf{Q}^{(N-1)}\mathbf{S}^{(N-1)}\mathbf{R}^{(N-1)}\tilde{\mathbf{I}}^{\mathrm{T}}. \tag{13}$$

Note, the $(N-1)$ compounds of $\mathbf{Q}$, $\mathbf{S}$ and $\mathbf{R}$ are of the respective orders of $N \times C_{N-1}^M$, $C_{N-1}^M \times C_{N-1}^M$ and $C_{N-1}^M \times N$, where

$$C_{N-1}^M = \frac{M!}{(N-1)!(R+1)!} \tag{14}$$

represents the set of unique combinations $(N-1)$ different columns which can be selected out of the M columns of the statics matrix. The product $\tilde{\mathbf{I}}\mathbf{Q}^{(N-1)}$ represents the row inversion (and sign alternation) of $\mathbf{Q}^{(N-1)}$, and $\mathbf{R}^{(N-1)}\tilde{\mathbf{I}}^{\mathrm{T}}$ is the column inversion (and sign alternation) of $\mathbf{R}^{(N-1)}$. Also, the $(N-1)$ compound of the modal stiffness matrix $\mathbf{S}$ is a diagonal matrix whose non-zero entries are all the combinations of products of $(N-1)$ stiffnesses in lexical order, that is, $12\ldots(N-1)N$, $12\ldots(N-1)(N+1),\ldots,(M-N+1)(M-N+2)\ldots M$.

Let $\mathbf{q}_n$ be the nth column of $\mathbf{Q}^{(N-1)}$ (also the nth row of $\mathbf{R}^{(N-1)}$) and let π_n be entry (n,n) of $\mathbf{S}^{(N-1)}$. It is easy to show that the inverse of $\mathbf{K}$ (13) can also be written as

$$\mathbf{K}^{-1} = \frac{\operatorname{adj}\mathbf{K}}{|\mathbf{K}|} = \frac{\sum_n \mathbf{q}_n\mathbf{q}_n^{\mathrm{T}}\pi_n}{\sum_k |\mathbf{K}|_k} \tag{15}$$

where the summation index n is carried over all the column indices of the $(N-1)$ compound of $\mathbf{Q}$. It will be recalled that the k indices in the denominator run over the statically determinate and stable subsets of $\mathbf{Q}$.

In summary, the inverse of the stiffness matrix, that is, the flexibility matrix of a structure, is equal to a sum of matrices divided by a sum of determinants. The determinants are those of the statically determinate stable subsets of the original structure. The physical interpretation for the matrices in the numerator is at this point unresolved.

In dynamic problems one often reduces the size of the stiffness matrix by performing a static condensation on degrees of freedom of secondary importance. Presently the explicit expression of the inverse of the stiffness matrix does not make allowance for such considerations.

5. THE INTERNAL FORCES

For the sake of completeness we will show in this section that the expression of the inverse of the stiffness matrix leads to the analytical expression of the internal forces, which was found independently. As stated in the introduction to this paper it was felt early on that the explicit equations of the internal forces originate from a proto-equation of the displacements much as the stress resultants in finite element analysis are obtained by pre-multiplying the displacements by the stress matrix. In equivalent structural terms, we will now show that eqns (1) result from the pre-multiplication of the displacements by the product of the natural stiffness matrix and the kinematics matrices

$$\mathbf{t} = \mathbf{SRu} = \mathbf{SRK}^{-1}\mathbf{p} \tag{16}$$

where $\mathbf{K}^{-1}$ is given by eqn (15). The denominator in eqn (15) is unaffected by this transformation and indeed we have already seen that it is already in the form given in eqn (1) if we assume, as was done earlier, that $B_k = |\mathbf{Q}_k|^2$. As a matter of fact, this implies that the B_ks are positive constants. What is left to be shown is that

$$|\mathbf{K}|\mathbf{t} = \sum_n \mathbf{SRq}_n\mathbf{q}_n^{\mathrm{T}}\pi_n\mathbf{p} \tag{17}$$

is identical to the numerator in the right-hand side of eqn (1).

Without loss of generality we will describe the process of matching the numerator in eqn (1) with the right-hand side of eqn (17) with the help of a 3×4 statics matrix ($N = 3$ nodal degrees of freedom, $M = 4$ elements). The $(N-1)$ compound of $\mathbf{Q}$ is a 3×6 matrix whose elements are all the 2×2 determinants which can be selected from $\mathbf{Q}$ in lexical (increasing) order. Let $(abcd)$ denote the determinants built on rows ab and columns cd of $\mathbf{Q}$. The reversed (and signed) 2-compound of $\mathbf{Q}$ becomes

$$\mathbf{\bar{I}Q}^{(2)} = \begin{bmatrix} (2312) & (2313) & (2314) & (2323) & (2324) & (2334) \\ -(1312) & -(1313) & -(1314) & -(1323) & -(1324) & -(1334) \\ (1212) & (1213) & (1214) & (1223) & (1224) & (1234) \end{bmatrix} \tag{18}$$

and $\mathbf{R}^{(2)}\mathbf{\bar{I}}^{\mathrm{T}}$ is the transpose of that matrix. In every column of this matrix the last two digits

$(\ldots cd)$ are constant and they uniquely define the column. The entries of a column cd of $\tilde{\mathbf{I}}\mathbf{Q}^{(2)}$ are the minors which can be constructed on columns c and d of $\mathbf{Q}$, but for the sign. In the matrix expansion of adj $\mathbf{K}$ in eqn (15), the matrix constructed on column cd of $\tilde{\mathbf{I}}\mathbf{Q}^{(2)}$ will be associated with π_{cd} where

$$\pi_{cd} = s_c s_d. \tag{19}$$

Consider, for instance, the first term on the right-hand side of eqn (17). It is composed of minors of column 12 of $\mathbf{Q}$,

$$(s_1 s_2 \mathbf{S})(\mathbf{R}\mathbf{q}_{12})(\mathbf{q}_{12}^T \mathbf{p}). \tag{20}$$

Note, π_{12} being a scalar was moved to the beginning of the expression. The net result of expression (20) is a 4×1 vector. To evaluate that vector we will consider the three parts of the above multiplication separately.

(a) $s_1 s_2 \mathbf{S}$—This term yields a diagonal matrix of products of stiffnesses $(s_1 s_2 s_1, s_1 s_2 s_2, s_1 s_2 s_3, s_1 s_2 s_4)$.

(b) $\mathbf{R}\mathbf{q}_{12}$—This is a vector whose entries are the sum of products of elements of $\mathbf{Q}^T$ and 2×2 signed minors of $\mathbf{Q}$. To evaluate these components, consider the 3×3 matrices obtained by taking column 12 and one of the remaining columns of $\mathbf{Q}$, that is matrices $\mathbf{Q}_{123}$ and $\mathbf{Q}_{124}$. If we take the adjoints of these matrices we will find the vector $\mathbf{q}_{12}$ in the last row of both adjoints. In other words, $\mathbf{q}_{12}$ is the third row of the adjoint of matrices $\mathbf{Q}_{123}$ and $\mathbf{Q}_{124}$. Recalling that the determinant of a matrix is equal to the sum of the elements of any row multiplied by their cofactors, it can be verified that entries 3 and 4 of $\mathbf{t}_1$ are the determinants $|\mathbf{Q}_{123}|$ and $|\mathbf{Q}_{124}|$. On the other hand, entries 1 and 2 of $\mathbf{q}_1$ are zero since the cofactors in $\mathbf{q}_1$ are alien to rows 1 and 2 of $\mathbf{Q}_{123}$, and expansions in terms of alien cofactors vanish identically.

(c) $\mathbf{q}_{12}^T \mathbf{p}$—Consider the statically determinate (and stable) substructure built on columns 123. If the external loads are applied to that structure alone the internal forces are given by

$$t_{123} = \frac{\text{adj}\,\mathbf{Q}}{|\mathbf{Q}_{123}|}\mathbf{p}. \tag{21}$$

Since $\mathbf{q}_{12}$ is equal to the last row of $\mathbf{Q}_{123}$, the last entry of t_{123}, that is the internal force in element 3, can be written as

$$t_{3,123} = \frac{1}{|\mathbf{Q}_{123}|}\mathbf{q}_1^T \mathbf{p}. \tag{22}$$

Applying the same logic to substructure 124 we obtain the following equations

$$\mathbf{q}_1^T \mathbf{p} = |\mathbf{Q}_{123}| t_{3,123} = |\mathbf{Q}_{124}| t_{4,124}. \tag{23}$$

Since $|\mathbf{S}_{123}| = s_1 s_2 s_3$ and $|\mathbf{S}_{124}| = s_1 s_2 s_4$ and using the information which was derived in the preceding paragraph, in conjunction with the product form of the stiffness matrix [eqn (6)], the expression in eqn (20) yields the vector

$$\{0 \quad 0 \quad |\mathbf{K}_{123}| t_{3,123} \quad |\mathbf{K}_{124}| t_{4,124}\}^T. \tag{24}$$

This simple, although painstaking, exercise gave the term resulting from the first column of the matrix in eqn (18). It will be recalled that it was constructed on columns 1 and 2 of $\mathbf{Q}$. Generalizing this result, one concludes that the contribution of any set of $(N-1)$ columns of $\mathbf{Q}$ to the right-hand side of eqn (17), is an M-vector whose component j has the following value: if column j is included in the set of $(N-1)$ columns then the jth component is identically zero; if j is an additional column, one considers the determinate substructure composed of the initial $(N-1)$ columns and column j, and the jth component is then equal to the determinant of the stiffness matrix of that substructure, multiplied by the internal force in element j of that substructure.

Repeating this procedure for the remaining five terms of the right-hand side of eqn (17) and grouping the results accordingly, one obtains the following expression for the internal forces in the case of the example

$$t_1 = \frac{1}{|\mathbf{K}|}(|\mathbf{K}_{123}|t_{1,123}+|\mathbf{K}_{124}|t_{1,124}+|\mathbf{K}_{134}|t_{1,134})$$

$$t_2 = \frac{1}{|\mathbf{K}|}(|\mathbf{K}_{123}|t_{2,123}+|\mathbf{K}_{124}|t_{2,124}+|\mathbf{K}_{234}|t_{2,234})$$

$$t_3 = \frac{1}{|\mathbf{K}|}(|\mathbf{K}_{123}|t_{3,123}+|\mathbf{K}_{134}|t_{3,134}+|\mathbf{K}_{234}|t_{3,234})$$

$$t_4 = \frac{1}{|\mathbf{K}|}(|\mathbf{K}_{124}|t_{4,124}+|\mathbf{K}_{134}|t_{4,134}+|\mathbf{K}_{234}|t_{4,234}) \tag{25}$$

with

$$|\mathbf{K}| = |\mathbf{K}_{123}|+|\mathbf{K}_{124}|+|\mathbf{K}_{134}|+|\mathbf{K}_{234}|. \tag{26}$$

Generalizing this result, the following remarkable expression emerges for the internal forces in a redundant structure

$$\mathbf{t} = \frac{1}{|\mathbf{K}|}\sum_k |\mathbf{K}_k|\mathbf{t}_k \tag{27}$$

where subscript k runs over all the statically determinate stable substructures which can be derived from the original structure, $|\mathbf{K}_k|$ is the determinant of the substructure and $\mathbf{t}_k$ are the corresponding internal forces (the forces in the missing elements are zero). It is clear this this equation is identical, although structurally more significant, to the expression in eqn (1) which was derived independently in Fuchs (1992a) for the case of a truss. Equation (15) for the inverse of the stiffness matrix and eqn (27) for the internal forces are valid for all linearly elastic structures.

6. THE EXPLICIT ANALYSIS OF A STAYED MAST

The stayed mast in Fig. 1 will help in visualizing the technique for generating the explicit analysis equations. The structure is composed of a vertical cantilever of length L and of uniform stiffnesses EA and EI where E is Young's modulus and A and I are respectively the cross-sectional area and the moment of inertia of the mast. The vertical deflection of the mast under the tip-load P is stiffened by three struts of stiffness EA_1, EA_2 and EA_3 which develop internal axial forces t_1, t_2 and t_3 respectively. Following Fuchs (1991) the mast is equivalent to three unimodal elements mounted in parallel, an extension element with axial force t_4, a shear element which carries the shear force $2t_5/L$ and the related differential moment t_5 and a moment element carrying the average moment t_6 of the mast. Consequently, the structure is composed of six unimodal elements as shown in Fig. 1, with stiffnesses [the diagonal entries of the stiffness matrix S in eqn (6)]

$$\mathbf{s} = \{EA_1/2L \quad \sqrt{2}EA_2/2L \quad \sqrt{3}EA_3/2L \quad EA/L \quad 3EI/L \quad EI/L\}^{\mathrm{T}}. \tag{28}$$

In the figure, the shear element is drawn with a hinge, since this element is structurally identical to a uniform beam with a hinge at mid-span. Similarly, the moment element is indicated by a shear release, since it behaves exactly as a uniform beam with a shear release anywhere along its span.

The three nodal equilibrium equations at the tip of the mast yield the following statics matrix

$$\mathbf{Q} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & \sqrt{2} & 1 & 0 & 4/L & 0 \\ 1 & \sqrt{2} & \sqrt{3} & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{bmatrix}. \tag{29}$$

From this redundant structure ($M = 6$, $N = 3$) one can derive $C_3^6 = 20$ [eqn (3)] combinations of statically determinate substructures out of which only 16 are stable. The statics sub-matrices $\mathbf{Q}_k$ are obtained by selecting combinations of three columns of $\mathbf{Q}$. Stable combinations are characterized by a non-zero determinant. Keeping in abeyance the expression of the inverse of the stiffness matrix, the internal forces can be written directly from eqn (1), and using $B_k = |\mathbf{Q}_k|^2$ to compute the coefficients. Table 1 shows the internal forces t_{ik} for the 16 stable substructures and Table 2 gives the corresponding B_k coefficients. The second column in both tables lists the three elements composing that substructure. An asterisk in Table 1 indicates that the force in that element is zero because the element is missing in the substructure.

With regard to the explicit form of the inverse of the stiffness matrix [eqn (15)], the approach is based on computing the $(N-1)$ or 2-compound of $\mathbf{Q}$. This is a 3×15 matrix ($C_2^6 = 15$, eqn 14) the columns of which are obtained by evaluating in turn the three determinants built on the combination of columns $12, 13, \ldots, 56$ of $\mathbf{Q}$ and storing them columnwise, and in lexical order, in $\mathbf{Q}^{(2)}$. Table 3 gives the compound matrix in transposed position, the second column of the table indicating the columns in $\mathbf{Q}$ from which the three determinants were computed. The reordered matrix $\tilde{\mathbf{I}}\mathbf{Q}^{(2)}$ used in the expression of adj $\mathbf{K}$ in eqn (13) is obtained by permuting rows 1 and 3 in $\mathbf{Q}^{(2)}$ and by multiplying row 2 by (-1). The generic matrices of the adjoint are obtained by selecting the 15 columns $\mathbf{q}_n$ of $\tilde{\mathbf{I}}\mathbf{Q}^{(2)}$ and forming the product $\mathbf{q}_n\mathbf{q}_n^T$ and multiplying the result by π_n. This yields the following expression for the adjoint of the stiffness matrix

Table 1. The internal forces in the substructures of the stayed mast

k	Element	t_1	t_2	t_3	t_4	t_5	t_6
—	123	—	—	—	—	—	—
—	124	—	—	—	—	—	—
1	125	$2/(\sqrt{3}-1)$	$-\sqrt{2}/(\sqrt{3}-1)$	*	*	0	*
2	126	$2/(\sqrt{3}-1)$	$-\sqrt{2}/(\sqrt{3}-1)$	*	*	*	0
—	134	—	—	—	—	—	—
3	135	$\sqrt{3}$	*	-1	*	0	*
4	136	$\sqrt{3}$	*	-1	*	*	0
5	145	$2/\sqrt{3}$	*	*	$-1/\sqrt{3}$	0	*
6	146	$2/\sqrt{3}$	*	*	$-1/\sqrt{3}$	*	0
7	156	0	*	*	*	$L/2$	$-L/2$
—	234	—	—	—	—	—	—
8	235	*	$\sqrt{6}/(\sqrt{3}-1)$	$-2/(\sqrt{3}-1)$	*	0	*
9	236	*	$\sqrt{6}/(\sqrt{3}-1)$	$-2/(\sqrt{3}-1)$	*	*	0
10	245	*	$\sqrt{2}$	*	-1	0	*
11	246	*	$\sqrt{2}$	*	-1	*	0
12	256	*	0	*	*	$L/2$	$-L/2$
13	345	*	*	2	$-\sqrt{3}$	0	*
14	346	*	*	2	$-\sqrt{3}$	*	0
15	356	*	*	0	*	$L/2$	$-L/2$
16	456	*	*	*	0	$L/2$	$-L/2$

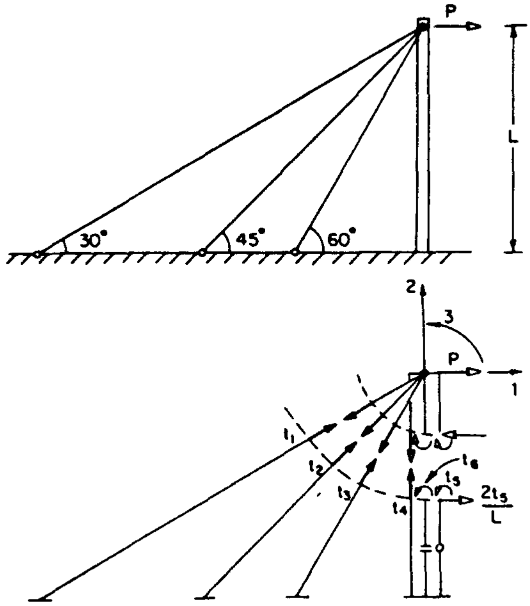

Fig. 1. The stayed mast and its unimodal components.

Table 2. The B_k constants of the stayed mast

k	Element	B_k	k	Element	B_k
1	125	$(\sqrt{3}-1)^2/8$	9	236	$(\sqrt{3}-1)^2/8$
2	126	$(\sqrt{3}-1)^2/8$	10	245	1/2
3	135	1/4	11	246	1/2
4	136	1/4	12	256	$2/L^2$
5	145	3/4	13	345	1/4
6	146	3/4	14	346	1/4
7	156	$1/L^2$	15	356	$3/L^2$
8	235	$(\sqrt{3}-1)^2/8$	16	456	$4/L^2$

Table 3. The 2-compound statics matrix of the stayed mast

Column	Columns in Q	Row 1	Row 2	Row 3
1	12	$\sqrt{2}(\sqrt{3}-1)/4$	0	0
2	13	1/2	0	0
3	14	$\sqrt{3}/2$	0	0
4	15	$-1/L$	$\sqrt{3}/2$	1/2
5	16	0	$\sqrt{3}/2$	1/2
6	23	$\sqrt{2}(\sqrt{3}-1)/4$	0	0
7	24	$\sqrt{2}/2$	0	0
8	25	$-\sqrt{2}/L$	$\sqrt{2}/2$	$\sqrt{2}/2$
9	26	0	$\sqrt{2}/2$	$\sqrt{2}/2$
10	34	1/2	0	0
11	35	$-\sqrt{3}/L$	1/2	$\sqrt{3}/2$
12	36	0	1/2	$\sqrt{3}/2$
13	45	$-2/L$	0	1
14	46	0	0	1
15	56	0	$2/L$	0

$$\operatorname{adj}\mathbf{K} = s_1 s_2 \begin{Bmatrix} 0 \\ 0 \\ \sqrt{2}(\sqrt{3}-1)/4 \end{Bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \sqrt{2}(\sqrt{3}-1)/4 \end{Bmatrix}^T + s_1 s_3 \begin{Bmatrix} 0 \\ 0 \\ 1/2 \end{Bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1/2 \end{Bmatrix}^T + \cdots + s_5 s_6 \begin{Bmatrix} 0 \\ -2/L \\ 0 \end{Bmatrix} \begin{Bmatrix} 0 \\ -2/L \\ 0 \end{Bmatrix}^T . \tag{30}$$

The inverse of $\mathbf{K}$ is obtained by dividing the adjoint by the determinant of the stiffness matrix.

It is easy to verify that this inverse, when used in eqn (16) recovers the internal forces given earlier. If we write the modal stiffnesses explicitly in terms of the cross-sectional areas and of the moment of inertia of the mast in these equations, we obtain the following explicit analysis results for the stayed mast

$$\begin{aligned} N_1 &= \frac{A_1}{|\mathbf{K}|}\left(\frac{\sqrt{2}(\sqrt{3}-1)}{4}A_2 + \tfrac{3}{4}A_3 + \sqrt{3}A\right) \\ N_2 &= \frac{A_2}{|\mathbf{K}|}\left(-\frac{\sqrt{3}-1}{4}A_1 + \frac{3(\sqrt{3}-1)}{4}A_3 + 2A\right) \\ N_3 &= \frac{A_3}{|\mathbf{K}|}\left(-\frac{\sqrt{3}}{4}A_1 - \frac{\sqrt{6}(\sqrt{3}-1)}{4}A_2 + \sqrt{3}A\right) \\ N &= \frac{A}{|\mathbf{K}|}\left(-\frac{\sqrt{3}}{2}A_1 - \sqrt{2}A_2 - \tfrac{3}{2}A_3\right) \\ M &= \frac{I/L}{|\mathbf{K}|}\left(\tfrac{3}{2}A_1 + \frac{3\sqrt{2}}{2}A_2 + \frac{9\sqrt{3}}{4}A_3 + 12A\right) \end{aligned} \tag{31}$$

with

$$|\mathbf{K}| = \frac{\sqrt{2}(\sqrt{3}-1)^2}{8}A_1A_2 + \frac{\sqrt{3}}{4}A_1A_3 + \tfrac{12}{8}A_1A + \tfrac{3}{2}A_1I/L^2 + \frac{\sqrt{6}(\sqrt{3}-1)^2}{8}A_2A_3 + \sqrt{2}A_2A + \frac{3}{\sqrt{2}}A_2I/L^2 + \frac{\sqrt{3}}{2}A_3A + \frac{9\sqrt{3}}{2}A_3I/L^2 + 12AI/L^2 \tag{32}$$

where $N_i = t_i$ $(i = 1, 2, 3)$ are the tensile forces in the struts and N and M $(= t_5 - t_6)$ are here respectively the tensile force and the root bending moment of the cantilever, the latter being defined positively for tensile stresses in the left outer fiber. Note the scale effect embedded in the $1/L$ and $1/L^2$ coefficients. For along and slender mast, the bending deformation vanishes from the equations and as expected the structure deforms in a pure truss mode.

7. CONCLUSIONS

This paper has for the first time presented the explicit inverse of the stiffness matrix of a linearly elastic structure. It is based on the property that the inverse of a non-singular matrix is equal to the adjoint of that matrix divided by the determinant of the matrix. The method for writing the inverse employs the congruent product form for the system stiffness matrix of the structure. The matrix is expressed as the product of the statics matrix, the unassembled element stiffness matrix in deformation coordinates and the kinematics matrix, in that order. For the general case of a redundant structure with M elements and N nodal degrees of freedom, the statics and kinematics matrices are rectangular matrices of order $N \times M$ and $M \times N$ respectively. It was indicated that by a proper transformation the unassembled element stiffness matrix is a diagonal matrix of order $M \times M$. This results from modeling the structure as an assemblage of unimodal elements.

Based on results obtained independently by Binet and Cauchy in 1812 it was shown that the determinant of the stiffness matrix is equal to the sum of the determinants of the stiffness matrices of all the statically determinate substructures which can be derived from the original structure. The expression for the adjoint of the stiffness matrix was obtained from the theorem on the product of compound (rectangular) matrices, also attributed to Binet and Cauchy. By a proper transformation, the $(N-1)$ compound of the stiffness matrix was related to the adjoint of the stiffness matrix, N being the number of degrees of freedom of the structure. This produced the analytic expression of the adjoint of the stiffness matrix as being the sum of generic matrices which are built with the columns of the transformed compound matrix. Having obtained the inverse of the stiffness matrix, and *ipso facto*, the explicit expression of the nodal displacements, the element internal forces were computed by premultiplying the displacements vector by the natural stiffness and kinematics matrices. The analytic expression of the internal forces in an elastic structure were found to be identical to results published by Fuchs in an earlier paper.

For the structural designer it is interesting to note that a nodal displacement is the ratio of two multilinear homogeneous polynomials in the unimodal stiffnesses of the structure, of order $(N-1)$ in the numerator and of order N in the denominator. Similarly, an internal force in a unimodal element is the ratio of two multilinear homogeneous polynomials in the unimodal stiffnesses of the structure, of order N, both in the numerator and in the denominator.

Unfortunately the analytic expressions, in their present form, seem to defy any practical application. The number of terms involved in the polynomials is simply immense. Consequently, the analytic inverse of the stiffness matrix can be employed for relatively modest structures only, or for structures with a low degree of static redundancy. For common engineering structures, one will rely on accepted numerical techniques.

Engineers, confronted with real world problems, occasionally derogate the concerns of fellow mathematicians. Redeeming Binet and Cauchy's seemingly esoteric work on compound matrices, laid down close to two centuries ago, was therefore a stimulating exercise in humility. In this spirit, it is hoped that the analytical expressions of the inverse of the stiffness matrix, the nodal displacements and the member forces of a linear elastic structure, will, in time, transcend the academic realm and find their way to the practice of structural engineering.

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